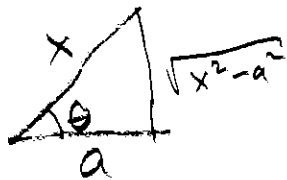


7.3 Trig. Substitution

Steps-by-step:

1. Substitute as directed below.
Simplify (should eliminate root)
Don't forget dx.
2. Use 7.2 methods for trig integrals.
3. Draw a triangle and return to x.

CASE	SUBSTITUTION
$a^2 - x^2$	$x = a \sin(\theta)$
$a^2 + x^2$	$x = a \tan(\theta)$
$x^2 - a^2$	$x = a \sec(\theta)$



Entry Task: Evaluate

$$\int \sqrt{9 + x^2} dx$$

$$x = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta d\theta$$

$$\int \sqrt{9 + 9 \tan^2 \theta} \cdot 3 \sec^2 \theta d\theta$$

$$9(1 + \tan^2(\theta))$$

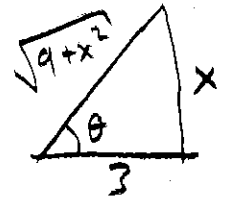
$$9 \int \sec \theta \cdot 3 \sec^2 \theta d\theta$$

$$9 \sec^3 \theta$$

$$9 \int \sec^3 \theta d\theta \quad \rightarrow \text{From TABLE}$$

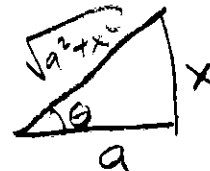
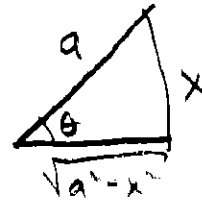
$$= \frac{9}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$$

$$= \frac{9}{2} \left(\frac{\sqrt{9+x^2}}{3} \cdot \frac{x}{3} + \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| \right) + C$$



$$= \left[\frac{1}{2} x \sqrt{9+x^2} + \frac{9}{2} \ln \left| \frac{\sqrt{9+x^2} + x}{3} \right| \right] + C$$

$$\tan \theta = \frac{x}{3}$$



Example:

$$\int \frac{\sqrt{x^2 - 16}}{x} dx$$

$$x = 4 \sec \theta$$

$$dx = 4 \sec \theta \tan \theta d\theta$$

$$\int \frac{\sqrt{16 \sec^2 \theta - 16}}{4 \sec \theta} 4 \sec \theta \tan \theta d\theta$$

$$\frac{\sqrt{16(\sec^2 \theta - 1)}}{4 \tan \theta}$$

$$\int 4 \tan \theta \cdot \tan \theta d\theta$$

$$4 \int \tan^2 \theta d\theta$$

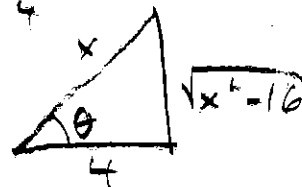
$$4 \int \sec^2 \theta - 1 d\theta$$

$$4(\tan \theta - \theta) + C$$

$$4 \tan \theta - 4\theta + C$$

$$\frac{4 \sqrt{x^2 - 16}}{4} - 4 \sec^{-1}\left(\frac{x}{4}\right) + C = \boxed{\sqrt{x^2 - 16} - 4 \sec^{-1}\left(\frac{x}{4}\right) + C}$$

$$\sec \theta = \frac{x}{4}$$



$$\theta = \sec^{-1}\left(\frac{x}{4}\right)$$

NOTE: IT IS NOT ACCEPTABLE

TO WRITE $\tan(\sec^{-1}(x/4))$

↑ ↑
NEVER WRITE
THIS.

Important Application

Area under a circle

$$\int \sqrt{4-x^2} dx$$

$$= \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta d\theta$$

$$= \int 2\cos\theta \cdot 2\cos\theta d\theta$$

$$= 4 \int \cos^2\theta d\theta$$

$$= 4 \int \frac{1}{2} (1 + \cos(2\theta)) d\theta$$

$$= 2 \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C$$

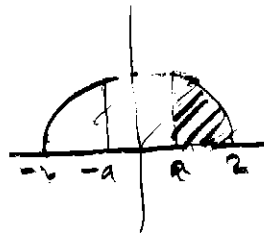
$$= 2\theta + \sin(2\theta) + C$$

??? ↓ HALF-ANGLE

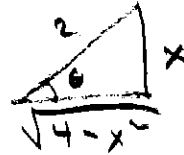
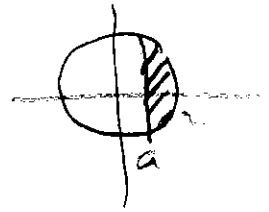
$$= 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + 2 \cdot \frac{x}{2} \frac{\sqrt{4-x^2}}{2} + C$$

$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} x \sqrt{4-x^2} + C}$$



$$x^2 + y^2 = 4$$



$$\sin\theta = \frac{x}{2}$$

CUT A CIRCLE INTO THIN DISCS?

$$2 \int_a^2 \sqrt{4-x^2} dx \stackrel{?}{=} \frac{1}{3} \pi (2)^2$$

$$2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2} x \sqrt{4-x^2} \Big|_a^2 \stackrel{?}{=} \frac{2}{3} \pi$$

$$\left(2 \cdot \frac{\pi}{2} + 0\right) - \left(2\sin^{-1}\left(\frac{a}{2}\right) + \frac{1}{2} a \sqrt{4-a^2}\right) \stackrel{?}{=} \frac{2}{3} \pi$$

$$\text{Solve } \Rightarrow a \approx 0.529864$$

Perfect Squares

$$(x + 3)^2 = x^2 + 6x + 9$$

SQUARED
↓
↑
HALF

$$(x - 5)^2 + 2 = x^2 - 10x + 25 + 2$$

Completing the Square

$$x^2 + 16x + 64 - 64 = (x + 8)^2 - 64$$

HALF
SQUARE
↓

$$3x^2 - 12x = 3(x^2 - 4x + 4 - 4) = 3((x - 2)^2 - 4)$$

HALF
↑

Given $\sqrt{ax^2 + bx + c}$

complete square!

- Factor out "a"
- Add/subtract half-middle squared

Example:

$$\int \sqrt{4x^2 - 32x + 100} dx$$

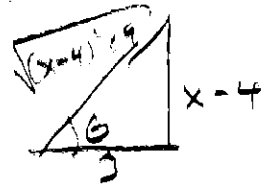
$$= \int \sqrt{4(x^2 - 8x + 25)} dx$$

$$= \frac{1}{2} \int \sqrt{x^2 - 8x + 25} dx$$

$$x^2 - 8x + 16 - 16 + 25 = (x - 4)^2 + 9$$

↑
SQUARE
9
HALF = 4

$$= 2 \int \sqrt{(x - 4)^2 + 9} dx$$



$$x - 4 = 3 \tan \theta$$

$$dx = 3 \sec^2 \theta d\theta$$

$$= 2 \int \frac{1}{3 \sec \theta} 3 \sec^2 \theta d\theta$$

$$= 2 \int \sec \theta d\theta$$

$$\sqrt{9 \tan^2 \theta + 9}$$

$$= 3 \sec \theta$$

$$= 3 \sec \theta$$

$$= 2 \ln |\sec \theta + \tan \theta| + C$$

$$= 2 \ln \left| \frac{\sqrt{(x-4)^2 + 9}}{3} + \frac{x-4}{3} \right| + C$$

Example:

$$\int \frac{x}{\sqrt{16 - 6x - x^2}} dx$$

$$\int \frac{x}{\sqrt{25 - (x+3)^2}} dx$$

$$\int \frac{5 \sin \theta - 3}{5 \cos \theta} 5 \cos \theta d\theta$$

$$= -5 \cos \theta - 3\theta + C$$

$$= -5 \frac{\sqrt{25 - (x+3)^2}}{5} - 3 \sin^{-1} \left(\frac{x+3}{5} \right) + C$$

$$16 - 6x - x^2$$
$$16 + 9 - 9 - 6x - x^2$$

↑ -3
HALF

$$25 - (9 + 6x + x^2)$$

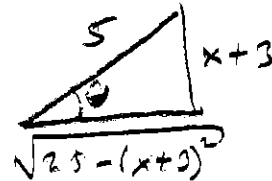
$$25 - (x+3)^2$$

$$x+3 = 5 \sin \theta \rightarrow x = 5 \sin \theta - 3$$

$$dx = 5 \cos \theta d\theta$$

$$\sqrt{25 - 25 \sin^2 \theta}$$

$$\sqrt{25 \cos^2 \theta}$$



$$\theta = \sin^{-1} \left(\frac{x+3}{5} \right)$$